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*Young Scientists:*

# Superpotentials of $\mathcal{N} = 1$ SUSY gauge theories

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**Abstract.** An introduction to recent advances in computing effective superpotentials of four dimensional $\mathcal{N} = 1$ SUSY gauge theories coupled to matter is presented. The correspondence with matrix models and two proofs of this are discussed and a novel derivation of the Veneziano-Yankielowicz superpotential for a pure gauge theory is given using the generalized Konishi anomaly. Based on a talk given at the School on Sub-Nuclear Physics, Erice, Sicily, September 2003.

## 1 Introduction

This work provides a brief introduction to recent advances in the study of four-dimensional gauge theories with the minimal amount (that is $\mathcal{N} = 1$) of supersymmetry. We consider general theories of this type, with arbitrary gauge group coupled to matter in an arbitrary representation, and with an arbitrary superpotential for the matter. The aim is to compute the effective superpotential for the gauge superfields obtained by integrating out the matter. Minimising this effective superpotential then yields the quantum vacua of the theory. In doing so, one discovers two remarkable things. The first remarkable thing is that the 'integrating out' part, which involves evaluating Feynman diagrams in superspace, reduces to the computation of planar diagrams in a matrix model. That is, the entire dependence on (super)space 'disappears'![1] The second remarkable thing is that, having done the perturbative computation in this way to $n$ loops, one finds (upon minimising the effective superpotential) that one has calculated the $n$-instanton correction to the vacuum structure. Thus, a *perturbative* computation leads to *non-perturbative* information about the physics!

Before we see how all of this comes about, let us make a rather simple, but nonetheless important remark about symmetries in physics in general. In any physical system, the presence of a symmetry (an invariance of the system) places a constraint on the possible dynamics of the system, and thus results in a simplification: in a sense, one of the reasons why we are so obsessed with symmetries in theoretical physics is because they simplify things. Ideally, one would like to consider systems with no symmetry at all. The set of such systems contains all more symmetric systems as a subset and so *general* statements made about *less* symmetric theories hold for all *more* symmetric theories. Unfortunately making such statements can be rather difficult.

Supersymmetries in gauge quantum field theories are no different. One can make general statements about gauge theories with no supersymmetry (for example the renormalisation group) and such statements are very powerful. What if we require our theory to have the *minimal* amount ($\mathcal{N} = 1$) of SUSY? Can we make even stronger general statements? Supersymmetry places rather strong constraints on a theory and we shall see (as we have claimed above) that it *is* possible to make stronger statements in this case. Moreover, because the statements apply to arbitrary $\mathcal{N} = 1$ theories, they apply equally to the subset of all theories with extended ($\mathcal{N} > 1$) SUSY, and it is interesting to see how results obtained previously in such theories (e.g. Seiberg-Witten duality in $\mathcal{N} = 2$ [1] and corollaries of Olive-Montonen duality in $\mathcal{N} = 4$ [2]) can be reproduced in the framework presented here [3].

We stress that although the methods apply to general minimally supersymmetric four-dimensional gauge theories, they do not tell us *everything* about such theories. A complete specification of the theory is given by the effective action; here one is only able to calculate the so-called $F$-terms (the effective superpotential) in the effective action. The Kähler, or $D$-terms (which lack the strong constraint of holomorphy) are not determined.

In the next section, we *sketch* two proofs [4, 5] of the gauge theory/matrix model correspondence (conjectured in [3]) and show how the superpotentials are calculated in each case. We then show that there are matter-independent (i.e. pure gauge) contributions to the effective superpotential which are undetermined. These contributions turn out to be non-perturbative and provide

[1] The disappearance is not a straightforward dimensional reduction however. If it were, the arguments presented here would hold in arbitrary dimensions; in fact they are specifically four-dimensional.

the bridge between the perturbative computation and the non-perturbative physics mentioned above. In Sect. 3 we show that these contributions can in fact be determined in this framework [6], and we do this.

## 2 The gauge theory/matrix model correspondence

We employ $\mathcal{N}=1$ superspace notations (see e.g. [7]). The gauge fields (and their superpartners) are written as components of a real vector superfield $V$; by acting with superspace covariant derivatives, one can form the analogue of the gauge field strength, $W_\alpha \sim \bar{D}^2 e^{-V} D_\alpha e^V$ and the gauge-invariant glueball chiral superfield $S \sim \mathrm{tr} W^\alpha W_\alpha$. The matter is represented by chiral superfields $\Phi$ with a tree-level matter superpotential in the action which is polynomial in the matter superfields[2]

$$\int d^4x d^2\theta W_{\mathrm{tree}} = \int d^4x d^2\theta g_k \Phi^k. \tag{1}$$

Here, the coefficients $g_k$ are called the tree-level matter couplings. We consider integrating out the matter $\Phi$ in some *background* glueball field $S$ to obtain an effective superpotential $W_{\mathrm{eff}}$, which depends on $S$, the $g_k$, and the gauge coupling (which we write in terms of the dimensionally-transmuted scale $\Lambda$).

It was claimed in the introduction that the perturbative computation of $W_{\mathrm{eff}}$ reduces to the evaluation of planar diagrams in a bosonic matrix model, and one might well ask how this can be demonstrated. Two proofs have appeared. The first [4] simply considers the contributing Feynman diagrams in superspace and shows that the momentum dependence of the bosons and their fermion superpartners cancels in all such diagrams. The only things left to consider are insertions of $S$, factors of $g_k$ coming from the vertices, and numerical symmetry factors. One can show that these can be obtained from planar diagrams of the matrix model

$$\exp \frac{F(S)}{g_s^2} = \int d\phi \exp \frac{W_{\mathrm{tree}}(\phi)}{g_s}, \tag{2}$$

where $\phi$ are $N' \times N'$ bosonic matrices and $S = g_s N'$. The restriction to planar diagrams is enforced by taking the 't Hooft limit: $N' \gg 1$ and $g_s \ll 1$, with $S$ fixed. The action of the matrix model is given by the tree-level matter superpotential $W_{\mathrm{tree}}$ with the matter superfields $\Phi$ replaced by bosonic matrices $\phi$.

To compute the perturbative computation to $W_{\mathrm{eff}}$ obtained by integrating out the matter (e.g. for gauge group $SU(N)$), one evaluates

$$N \frac{\partial F(S)}{\partial S}, \tag{3}$$

where $F(S)$ is the perturbative free energy of the matrix model in the planar limit.

The second proof [5] is rather different. One considers the effect of general chiral changes of variables $\delta\Phi = \epsilon f(\Phi, W_\alpha)$ in the path integral. These lead to anomalous Ward identities generalizing the Konishi anomaly [8, 9]. For example, the variation $\delta\Phi = \epsilon\Phi'(\Phi)$ yields

$$\left\langle \Phi' \frac{\partial W_{\mathrm{tree}}}{\partial \Phi} - S \frac{\partial \Phi'}{\partial \Phi} \right\rangle = 0. \tag{4}$$

From the general chiral change of variables specified by $f$, one obtains a complete set of anomalous Ward identities for the chiral matter fields, and one can show that these are in one-to-one correspondence with the complete set of Ward identities in the matrix model (which, since the matrix model partition function (2) is just an integral, correspond to integration by parts identities). This establishes the correspondence between the SUSY gauge theory and the bosonic matrix model.

Having established the correspondence, one can go on and calculate the effective superpotential for any given theory. To do this, one needs to solve the complete set of Ward identities to obtain the expectation values $\langle \Phi^k \rangle$ appearing in the tree-level matter superpotential in terms of the background glueball superfield $S$ and the couplings $g_k$. The effective superpotential can then be determined from the partial differential equations

$$\frac{\partial W_{\mathrm{eff}}}{\partial g_k} = \langle \Phi^k \rangle, \tag{5}$$

which follow from standard supersymmetry and holomorphy arguments.

We note that these partial differential equations only specify the effective superpotential up to a term which is independent of the matter couplings $g_k$, but which may depend on both $S$ and the gauge coupling scale $\Lambda$. This term must contain any contribution to the effective superpotential coming from the *pure* gauge theory *without* matter. So let us ask the question: is there a pure gauge theory contribution? It turns out that there is, as was shown many years ago by Veneziano and Yankielowicz [10, 11] using the $U(1)_R$ symmetry of the pure gauge theory. For the gauge group $SU(N)$, for example, the pure gauge theory superpotential is

$$W_{\mathrm{eff}}(S, \Lambda) = N \left( -S \log \frac{S}{\Lambda^3} + S \right). \tag{6}$$

Such terms are non-perturbative. One way to see this is to minimise $W_{\mathrm{eff}}$ with respect to $S$. This reproduces the vacuum condensate $S^N = \Lambda^{3N}$, due to (non-perturbative) instantons [12]. In the next section, we show how such terms can be derived using the generalized Konishi anomaly in the presence of matter discussed above [6]. This then

[2] Only polynomials of degree three or less are renormalizable. However, since we claim that the computation of the effective superpotential reduces to a matrix model, the results must be independent of the momenta and any momentum cutoff. The results are thus independent of the UV completion of the theory and one is free to consider 'non-renormalizable' tree-level superpotentials.

renders the above approach self-contained, as well as providing an independent derivation of the Veneziano-Yankielowicz terms.

Before doing so, one might ask where the missing terms are hidden in the perturbative approach using Feynman diagrams. One might assume that they correspond to diagrams with gauge superfields in the loops; this is not really correct, since the missing terms are non-perturbative and thus cannot show up in any diagram. Intriguingly, these terms can be generated by the measure of the matrix model (i.e. the volume of the gauge group) [13, 3], though it is not at all clear why.

## 3 Pure gauge terms

In order to derive the pure gauge theory contributions, we determine the effective superpotential in the case where the matter sector consists of $F$ flavours of 'quarks' transforming in the fundamental representation of the gauge group, which we take to be $SU(N)$ (though the argument can be applied to any classical Lie group). Furthermore, we choose a tree-level superpotential in which the quarks can have either zero or non-zero classical expectation values at the minima. If a quark has a non-zero *vev*, then since the quarks transform non-trivially under the gauge group, the gauge group must be spontaneously broken via the Higgs mechanism. By putting each of the $F$ quarks at zero or non-zero minima, we can *engineer* the gauge symmetry breaking such that the unbroken gauge group is anything from $SU(N)$ down to $SU(N-F)$. We then solve the Konishi anomaly Ward identities (4) and the resulting partial differential equations (5), determining the effective superpotential in each vacuum, up to a constant term (by 'constant' we mean 'independent of the tree-level matter couplings').

The tree-level matter couplings are free parameters in the theory. We vary them such that both the quark masses and the Higgs *vevs* (which determine the masses of the massive gauge bosons) become large. In that limit, the massive matter decouples from the unbroken low energy gauge group, and the effective superpotential contains a sum of contributions from the decoupled matter and the low energy gauge group. Once we have identified the contribution of the massive matter and discarded it, we are left with the superpotential of the low energy gauge group. This includes the constant term.

Now any two distinct vacua have different unbroken gauge groups, but the same constant term. If we subtract the two superpotentials (with the massive matter discarded), the constant cancels and we are left with a difference equation for the pure gauge theory superpotential. The solution to this difference equation yields precisely the Veneziano-Yankielowicz terms (6). To determine the constant term in any theory, one then demands that $W_{\mathrm{eff}}(S, g_k, \Lambda)$ reproduces the correct decoupled contributions of the unbroken gauge group and massive matter in any vacuum in the massive limit [14]. Incidentally, the fact that the matching in one vacuum correctly reproduces the superpotential in all vacua justifies *a posteriori* the assumption that the constant term is the same for each vacuum branch.

Having explained the argument, let us now carry it out. Since quarks are Dirac fermions and chiral supermultiplets contain Weyl fermions, we represent $F$ flavours of quarks by $F$ chiral superfields $Q_i$ transforming in the fundamental representation of $SU(N)$ and a further $F$ chiral superfields $\tilde{Q}^j$ transforming in the anti-fundamental representation. The tree-level matter superpotential is written in terms of the gauge invariant mesons $M_i^j = Q_i \tilde{Q}^j$ as

$$W_{\mathrm{tree}} = m \mathrm{tr} M - \lambda \mathrm{tr} M^2. \tag{7}$$

The classical vacua are then

$$m M_i^j - 2\lambda M_i^k M_k^j = 0, \tag{8}$$

with $F_-$ eigenvalues at $M_i^i = 0$ and $F_+ = F - F_-$ eigenvalues at $M_i^i = m/2\lambda$. If $M_i^i$ has a non-zero vev, then so have $Q_i$ and $\tilde{Q}^i$, and the gauge symmetry is broken. The low energy gauge group is thus broken down to $SU(N-F_+)$. The quantum theory has the Konishi anomaly and the classical vacua are modified to (4)

$$m\langle M_i^j \rangle - 2\lambda \langle M_i^k M_k^j \rangle = \delta_i^j S, \tag{9}$$

with $F_\pm$ eigenvalues at

$$\langle M_i^i \rangle = \frac{m}{4\lambda} \left( 1 \pm \sqrt{1 - \frac{8\lambda S}{m^2}} \right). \tag{10}$$

The partial differential equations following from holomorphy and supersymmetry are [5]

$$\begin{aligned} \frac{\partial W_{\mathrm{eff}}}{\partial m} &= \langle \mathrm{tr} M \rangle, \\ \frac{\partial W_{\mathrm{eff}}}{\partial \lambda} &= -\langle \mathrm{tr} M^2 \rangle. \end{aligned} \tag{11}$$

We shall not write here the expression for the effective superpotential $W_{\mathrm{eff}}$ which is obtained by integrating these equations (it is rather cumbersome). Taking the limit of $W_{\mathrm{eff}}$ in which the quark mass $m$ and Higgs vev $\sqrt{m/2\lambda}$ become large and subtracting the superpotentials for the vacua in which the number of Higgsed quarks is $F_{1,2}$, one obtains

$$\begin{aligned} &W_{\mathrm{eff},1} - W_{\mathrm{eff},2} \to \\ &\quad (F_1 - F_2)\frac{m^2}{4\lambda} + (F_1 - F_2)\left[ S \log \frac{S}{m^2/2\lambda} - S \right]. \end{aligned} \tag{12}$$

The first term represents the decoupled matter: it is given (according to the non-renormalization theorem) by the classical expectation value of $W_{\mathrm{tree}}$. The second term must therefore represent the contribution of the low energy pure gauge group $SU(N - F_{1,2})$.[3] It seems peculiar that what

[3] There is a subtlety here: the glueball superfield $S$ includes the massive gauge bosons, which should be integrated out by replacing them with their *vevs*. However, in the decoupled limit, the massive gauge bosons have zero *vevs*, so the field $S$ is equivalent to the glueball superfield of the low energy gauge group once the massive gauge bosons have been integrated out.

we have identified as the superpotential of the low energy gauge group contains the matter couplings $m$ and $\lambda$. However, these are precisely the factors needed to convert the $SU(N)$ gauge coupling scale $\Lambda$ to the $SU(N-F_{1,2})$ scales $\Lambda_{1,2}$ via the scale-matching relation

$$\Lambda_1^{3(N-F_1)}\left(\frac{m^2}{2\lambda}\right)^{F_1} = \Lambda^{3N-F} m^F = \Lambda_2^{3(N-F_2)}\left(\frac{m^2}{2\lambda}\right)^{F_2}. \tag{13}$$

This relation comes from requiring that the coupling constants of the high energy theory (with dynamic matter) and the low energy theory (with matter integrated out) match at the Higgs and quark mass scales (see *e.g.* [15]). Replacing the matter couplings by the appropriate gauge coupling scales in this way (and discarding the massive matter) leads to the difference equation

$$W_{\text{eff},1} - W_{\text{eff},2} = (N-F_1)\left(-S\log\tfrac{S}{\Lambda_1^3}+S\right) - (N-F_2)\left(-S\log\tfrac{S}{\Lambda_2^3}+S\right), \tag{14}$$

with solution

$$W_{\text{eff}}(S,\Lambda) = N\left(-S\log\frac{S}{\Lambda^3}+S\right) + f(S). \tag{15}$$

Here, $f(S)$ is an arbitrary function of $S$ alone; it is independent of all other parameters. On dimensional grounds, $f(S)\propto S$ and one sees that the ambiguity in $f$ (which can be re-written as a pure number multiplying $\Lambda^{3N}$) corresponds to the freedom to choose a renormalisation group scheme [15].

## 4 Discussion

The methods summarised above provide a very powerful framework in which to study gauge theories with $\mathcal{N}=1$ SUSY, and it is certainly of interest to go on and study the vacuum structure and phases of specific models.

More general extensions to this work include the question of whether similar results hold in dimensions other than four [16], the extension to supergravity (rather than supergauge) backgrounds [3,17,18,19,20] and whether dynamical breaking of supersymmetry may be studied in this framework.

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